Asymptotics Methods to Compute Electromagnetic Fields

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***Abstract*— We provide a rigorous asymptotic method to compute the electromagnetic fields in domains with thin layer. With this method the influency of the membrane on the inner field is replaced by an approximated boundary condition, while in the thin layer, the approximated field is explicitely known. We give error estimates, which validate our asymptotic.**

I. Introduction

In this paper we present a rigorous asymptotic method to compute electromagnetic fields in domains with thin layer. The main idea of the asymptotics is to replace the thin membrane either by an appropriate boundary condition or by a transmission condition, depending on the problem. With our approach, the influence of the membrane is known but we do not have to mesh it. Moreover we give error estimates between the total field and our approximation. In this proceeding we describe how to build an approximated boundary condition, the case of approximated transmission conditions will be presented in the extended version.

II. Heuristics of the asymptotics

In this section, we present the heuristics of the asymptotics on a simple example: the dielectric formulation (1). Let Ω be a domain composed of a material $\mathcal{O}$ surrounded by a thin layer $\mathcal{O}_h$ with thickness h. Let γ_0 be the complex permittivity of $\mathcal{O}$ and γ_1 those of $\mathcal{O}_h$. For sake of simplicity, we suppose that γ_0 and γ_1 are both constant, but our results are valid even if γ_0 is not a constant.

We denote by γ the function equal to γ_0 in $\mathcal{O}$ and γ_1 in $\mathcal{O}_h$. Consider the quasistatic approximation of voltage potentials:

$$\begin{cases} \nabla\cdot(\gamma\nabla V)=0 \\ \dfrac{\partial V}{\partial n}=\phi, \end{cases} \tag{1}$$

where ϕ is a boundary condition as regular as needed. The idea is to write Laplace operator in local coordinate so that the thin parameter h appears explicitly. Denote by $\Delta_{\eta,\theta}$ Laplace operator written in (η,θ)-coordinates. We have:

$$\begin{aligned}\Delta_{\eta,\theta} =& \frac{1}{1+h\eta\kappa}\partial_\eta\left(\frac{1+h\eta\kappa}{h}\partial_\eta\right) \\ &+\frac{1}{1+h\eta\kappa}\partial_\theta\left(\frac{h}{1+h\eta\kappa}\partial_\theta\right),\end{aligned} \tag{2}$$

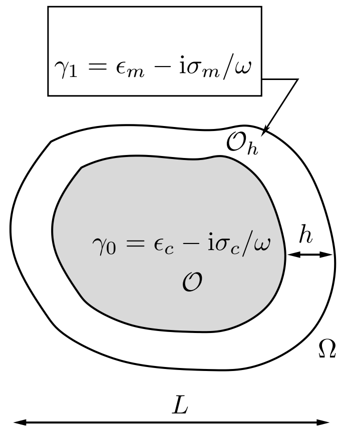

Fig. 1. The domain Ω. ϵ and σ are the permittivities and the conductivities of the domain, ω is the pulsation and L is the diameter of Ω.

where (η,θ) belongs to $(0,1)\times\mathbb{R}/2\pi\mathbb{Z}$. We suppose that V can be written as follows:

$$V=V_0+hV_1+\cdots.$$

We replace V by its developpement in (2) and we identify the terms of the same power of h. We refer to [4] or [3] for more details.

III. Results

In this part we give the first two terms of the asymptotic expansion of the solution V of Problem (1). In the extended version, we will give the results for Helmholtz equation and we will also consider the case where the domain Ω is embedded in an ambient medium.

A. Approximated Boundary Condition on $\partial\mathcal{O}$

- Order 0.
  V_0^c satisfies the following problem:
$$\begin{cases} \nabla\cdot(\gamma_0\nabla V_0^c)=0, \text{ in } \mathcal{O}, \\ \partial_n V_0^c=(\gamma_1/\gamma_0)f \text{ on } \partial\mathcal{O}. \end{cases} \tag{3}$$
- Order 1.
  V_1^c satisfies:
$$\begin{cases} \nabla\cdot(\gamma_0\nabla V_1^c)=0, \text{ in } \mathcal{O}, \\ \partial_n V_1^c=(\gamma_1/\gamma_0)\partial_t^2 V_0^c \text{ on } \partial\mathcal{O}_c. \end{cases} \tag{4}$$

Denote by $V^c_{app} = V^c_0 + hV^c_1$. We have the following estimate:

$$\|V^c - V^c_{app}\|_{H^1(\mathcal{O})} \leq C\frac{\gamma_1}{\gamma_0}h^2\|\phi\|_{H^4(\partial\mathcal{O}_h)}. \qquad (5)$$

We emphasize that the constant C depends uniquely on geometric parameters of the materials. Inequality (5) estimates the error between V^c and V^c_{app} and also the error of their gradient. Observe that V^c_{app} satisfies

$$\begin{cases} \nabla \cdot \left(\gamma_0 \nabla V^c_{app}\right) = 0, \\ \partial_n V^c_{app} = (\gamma_1/\gamma_0) f + (\gamma_1/\gamma_0)\left(h\partial^2_t V^c_{app} - h^2 \partial^2_t V^c_1\right). \end{cases} \qquad (6)$$

Let $\tilde{V}^c_{app}$ be the solution to

$$\begin{cases} \nabla \cdot \left(\gamma_0 \nabla \tilde{V}^c_{app}\right) = 0, \ \text{ in } \mathcal{O}, \\ \partial_n \tilde{V}^c_{app} - (\gamma_1/\gamma_0) h \partial^2_t \tilde{V}^c_{app} = (\gamma_1/\gamma_0) f \ \text{ on } \partial\mathcal{O}_c, \end{cases} \qquad (7)$$

then, using (5) we have:

$$\|V^c - \tilde{V}^c_{app}\|_{H^1(\mathcal{O})} \leq C\frac{\gamma_1}{\gamma_0}h^2\|\phi\|_{H^4(\partial\mathcal{O}_h)}. \qquad (8)$$

The boundary condition satisfied by $\tilde{V}^c_{app}$ is called approximated boundary condition. Observe that this approximated boundary condition is exactly the same as boundary condition (10) of Krähenbühl and Muller [2]. Here, we validate the result by an error estimate. Emphasize that we can perform asymptotic expansion at any order $n \geq 0$; in the extended version, we will give general expression of the approximated boundary condition.

In the thin layer, we have explicit formulae of V written in local coordinates:

$$V^m_0 = V^c_0|_{\partial\mathcal{O}_c}, \quad V^m_1 = \eta\varphi + V^c_1|_{\partial\mathcal{O}_c}.$$

Denote by $V^m_{app} = V^m_0 + hV^m_1$. We have the following estimate:

$$\|V^m - V^m_{app}\|_{H^1(\mathcal{O}_h)} \leq Ch^{3/2}\|\phi\|_{H^4(\partial\mathcal{O}_h)}. \qquad (9)$$

B. Numerical simulations

We perform calculus with GetDP [1] when Ω is an elongated cell. In Fig. 2 we present the steady state potentials when the thin layer is insulating.

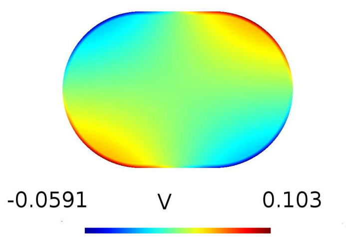

Fig. 2. Steady state potentials in an elongated cell with insulating membrane.

Asymptotics (3) et (7) give the approximated potential in $\mathcal{O}$ without meshing the membrane.

In Fig. 3 we draw the error made by our asymptotics with respect to the thickness h for a slightly conductive membrane. Observe for instance that if $h = 5.10^{-3}$ the error made by our

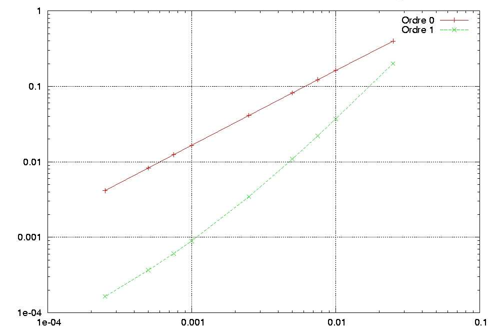

Fig. 3. Error made by our asymptotics.

method is around 10% at the order 0 and 1% at the order 1.

IV. CONCLUSION

We have presented in a simple case how to build rigorously and with error estimate an approximated boundary condition at the order 1 for the solution of steady state voltage potentials in a domain with thin membrane. With our method we do not have to mesh the membrane: the approximated field inside the thin layer is known explicitly, and the cytoplasmic field is solution to the dielectric formulation with the approximated boundary condition. At the order 1, we recover the boundary condition of Krähenbühl and Muller [2], and we give error estimate, moreover, by our method, we can build approximated boundary condition at any order, if the domain and the boundary data are enough regular. This means that we could build solutions, which approach the total potential with an error in h^n, for n as big as desired. This method has been generalized to Helmholtz equation [5] and we except that we can apply it to the vector wave equation. If the cell is embbed in an ambient medium, we obtain appropriate transmission conditions on the boundary of the cytoplasm [5]. We emphasize that in the thin layer, the approximated potential is explicitely known in local coordinates.

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